Dynamics on tori.

Recall that parabolic Riemann surfaces are conformally isomorphic to quotients of $\mathbb{C}$ by suitable groups of translations, and we gets $\mathbb{C}$, $\mathbb{C}^* \simeq \mathbb{C}/\mathbb{Z}$, or $\mathbb{C}/\Lambda$, where $\Lambda \subset \mathbb{C}$ is a rank 2 lattice, generated by $\langle 1, \tau \rangle$, $\tau \in \mathbb{C} \setminus \mathbb{R}$.

We focus on the latter case, $\mathbb{T} = \mathbb{C}/\Lambda$.

We start with some examples: affine maps.

Consider the affine map $F_{\alpha,\beta} : \mathbb{C} \to \mathbb{C}$, $F_{\alpha,\beta}(z) = \alpha z + \beta$.

If $\alpha = 0$, the map is constant and clearly induces a constant map $f_{0,\beta} : \mathbb{T} \circlearrowleft$ on the quotient. We assume $\alpha \neq 0$.

Then $F_{\alpha,\beta}$ induces a map $f_{\alpha,\beta} : \mathbb{C}/\Lambda \circlearrowleft$ if and only if:

$\alpha(z+1) + \beta - (\alpha z + \beta) = \alpha \in \Lambda$, and

$\alpha(z+\tau) + \beta - (\alpha z + \beta) = \alpha\tau \in \Lambda$.

If $\alpha = 1$, clearly $F_{1,\beta}$ passes to the quotient to an automorphism $f_{1,\beta}$ of $\mathbb{T}$, the inverse given by $f_{1,-\beta}$. We will assume $\alpha \neq 1$.

In this case, $F_{\alpha,\beta}$ is conjugate to $F_{\alpha,0} =: F_\alpha : z \mapsto \alpha z$, through the conjugacy map $\Phi(z) = z + \frac{\beta}{1-\alpha}$.

Being $\Phi' = 1$, this map passes to the quotient and defines a conjugacy between $f_{\alpha,\beta}$ and f_α. We will hence assume $\beta = 0$.

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Notice that $\forall \alpha \in \mathbb{Z}$ leaves Λ invariant. Hence any torus admits a multiplication by an integer.

We want to understand if other multiplication are possible.

Def: A torus $\mathbb{T} = \mathbb{C}/\Lambda$ has complex multiplication (CM) if $\exists \alpha \in \mathbb{C} \setminus \mathbb{R}$ such that $\alpha \Lambda \subseteq \Lambda$.

Now, if $\alpha\Lambda \subseteq \Lambda$, then we have $(\Leftrightarrow)$ $\exists \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{=A_\alpha} \in \mathrm{Mat}(2\times 2, \mathbb{Z})$, $\det A \neq 0$, s.t.

$$\begin{cases} \alpha = a + b\tau \\ \alpha\tau = c + d\tau \end{cases} \quad \text{i.e. } \alpha \cdot \begin{pmatrix} 1 \\ \tau \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ \tau \end{pmatrix}.$$

Hence the multiplication by α defines a $\mathbb{R}$-linear map L_α on $\mathbb{C}$, represented by A_α on the basis $(1, \tau)$.

Hence it must satisfy $L_\alpha^2 - \underset{p}{\underbrace{(a+d)}} L_\alpha + \underset{q}{\underbrace{(ad-bc)}} I = 0$.

Evaluating at 1, we get $\alpha^2 - p\alpha + q = 0$ $(*)$

(Rem: $\alpha \in \mathbb{Z}$ corresponds to the case of $A_\alpha = \alpha I$, and $(*)$ is trivially satisfied).

If $\alpha \notin \mathbb{Z}$ $(\Rightarrow \alpha \notin \mathbb{R})$, then $\bar{\alpha}$ must be another solution of $(*)$ (being $p, q \in \mathbb{R}$), and we get $\alpha\bar{\alpha} = |\alpha|^2 = q \in \mathbb{N}^*$.

Moreover, we have $p^2 - 4q < 0$, hence $p^2 < 4q$.

In particular, for any q, there exist finitely many p, ~~for which~~ and hence finitely many α that can preserve a lattice.

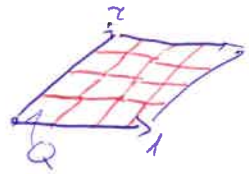
Prop: q is the degree of f_α. That is, $\forall w \in \mathbb{T}$, $\# f_\alpha^{-1}(w) = q$.

Proof: f_α is a holomorphic map with non-vanishing derivative, between

Compact surfaces. Hence it is a covering map with finite fibers.
Denote by D the cardinality of $f^{-1}(w)$ (for any w, being this number constant). We want to show that $D=q$.

Since $q=\det A$, f_α sends polygons (squares) of area S to polygons (parallelograms) of area qS.

We can cover $\mathbb{T}$ with finitely many (say N^2) parallelograms evenly covered with respect to the covering map f_α, intersecting only on the boundary (of area 0).



Then, if we normalize so that $\mathrm{Area}(\mathbb{T})=1$, we have:

$$1=\sum_{j=1}^{N^2}\frac{1}{N^2}=\sum_{j=1}^{N^2}Q_j=\sum_{j=1}^{N^2}d\cdot\frac{Q_j}{q}=\frac{d}{q}\cdot\sum Q_j=\frac{d}{q}\qquad\Rightarrow d=q.$$ □

Corollary. f_α has exactly $|\alpha-1|^2$ fixed points.

Proof: $\alpha z \hat{=} z \iff (\alpha-1)z \hat{=} 0$. But we know that this equation has $|\alpha-1|^2$ preimages.

In particular, $|\alpha|^2\in\mathbb{N}$ and $|\alpha-1|^2\in\mathbb{N}$.

Case $q=1$. In this case, f_α is an automorphism.

In this case we get: $p^2<4 \xrightarrow[\text{Computation}]{}$ α must be a m-th root of unity, with $m=3$ $(p=-1)$, 4 $(p=0)$, 6 $(p=1)$, plus the cases $m=1,2$ for integers.

In this case it is easy to check that $J(f_\alpha)=\emptyset$ (in fact, $f_\alpha^{12}=\mathrm{Id}$).

Similarly for translations $f_{1,\beta}$ $J(f_{1,\beta})=\emptyset$ (For any sequence $n\beta$ mod Λ, we can extract a convergent subsequence by compactness, and the corresponding translations converge uniformly).

Case $q \geq 2$.

In this case, f_α has $|\alpha^n - 1|^2$ n-periodic points.

The value $|\alpha^n - 1|^2 \to \infty$ being $|\alpha| > 1$. Moreover, periodic points have multiplier α^m, which is > 1 in modulus. Hence they belong to the Julia set.

It is easy to see that the set of periodic points is also dense in $\mathbb{T}$, hence $J(f_\alpha) = \mathbb{T}$ in this case.

For preperiodic points it suffices to show that $\forall\, U$ open, $f_\alpha^n(U) = \mathbb{T}$ for $n >> 0$. In particular U contains preperiodic points.

We now show that the example we studied is in fact the general situation.

Theorem: Any holomorphic map $f : \mathbb{T} \circlearrowleft$ is induced by an affine map $F : z \mapsto \alpha z + \beta$, $F : \mathbb{C} \circlearrowleft$, (so that $\alpha \Lambda \subseteq \Lambda$).

Proof. Any $f : \mathbb{T} \circlearrowleft$ lifts to a map $F : \mathbb{C} \circlearrowleft$, being $\mathbb{C}$ simply connected. The map f satisfies: $\exists\, \alpha \in \Lambda$ $\exists \gamma$ so that

$F(z+1) - F(z) = \alpha \quad \forall z \in \mathbb{C}$. (two translates by $1, \tau$ must be in Λ).

$F(z+\tau) - F(z) = \gamma \quad \forall z \in \mathbb{C}$.

Set $G(z) = F(z) - \alpha z$, so that $G(z+1) = F(z+1) - \alpha z - \alpha = F(z) - \alpha z = G(z)$.

Then $G(z+\tau) = F(z+\tau) - \alpha(z+\tau) = F(z) + \gamma - \alpha z - \alpha\tau = G(z) + \gamma - \alpha\tau$.

We claim that $G \equiv \beta$ constant.

In fact G induces a map $g : \mathbb{T} \to \mathbb{C}/(\gamma - \alpha\tau)\mathbb{Z} =: Y$.

Notice that $\begin{cases} \gamma = \alpha\tau \Rightarrow Y \simeq \mathbb{C} \\ \gamma \neq \alpha\tau \Rightarrow Y \simeq \mathbb{C}^* \end{cases}$ (In both cases Y is non compact)

Theorem (Riemann-Hurwitz formula). Let $f: Y \to X$ be a ramified covering map between compact Riemann surfaces. Then: ⇕ holomorphic non constant.

$$d\chi(X) - \chi(Y) = \sum_{y \in Y} (e_y - 1), \quad \text{where:}$$

d is the degree of f ($=$ # preimages of a generic point x = non ~~ramified~~ one)

χ denotes the Euler characteristic: $\chi(S) = 2 - 2g(S)$ g = genus.

$\forall y \in Y$: e_y denotes the ramification index at y: i.e. locally at y the map f is given by $z \mapsto z^{e_y}$.

Idea of the proof: Work with triangulations. If f is a covering map, the right hand side (RHS) is zero, and any triangle in X gives d triangles in Y. A ramification point $x \in X$ means that we identify $\forall y, f(y) = x$ e_y points (vertices of some triangles) to the same point, hence it contributes with $e_y - 1$ to the computation of the Euler characteristic.

Rem: the formula works also for non-compact surfaces as far as f is proper (hence a covering map)

Corollary: In the previous construction, $X_\infty \simeq \hat{\mathbb{C}}$.

Proof: if $m = 2$, we have 4 ramified points of index 2, hence:

(RH) $4 = 2 \cdot \chi(X) - 0 \Rightarrow \chi(X) = 2$ and $X \simeq \hat{\mathbb{C}}$.

If $m = 3$, we have 3 ramified points of index 3, and $6 = 3\chi(X)$ OK.

If $m = 4$, we have 2 ramified points of index 4 (the fixed points).

We also have a 2-cycle, which corresponds to two branching points of index 2 (projecting to the same ramification point).

RH: $2 \cdot 3 + 2 \cdot 1 = 4\,\chi(X) \Rightarrow \chi(X) = 2$ OK

Finally for $m = 6$, we get 1 fixed point, 1 2-cycle and 1 3-cycle:

RH: $5 + 2 \cdot 2 + 3 \cdot 1 = 12 = 6 \cdot \chi(X) \Rightarrow \chi(X) = 2$ OK

□

Up to this isomorphism, $g: \mathbb{T} \to \mathbb{C}$ is holomorphic and bounded ($\mathbb{T}$ is compact), and $|g|$ admits a local max, hence it is constant by the max principle. Hence G is itself constant, and we are done $\square$

Lattès Maps

From affine maps on complex tori we can construct special rational maps on the Riemann Sphere, as follows.

Consider $\mathbb{T} = \mathbb{C}/\Lambda$ a torus. We have seen that automorphisms of the torus are induced either by translations, or by linear maps $z \xrightarrow{L} \omega z$ with ω a 1, 2, 3, 4, or 6-th root of unity.

For translation, either the map f has finite order and the quotient $T/_{(z+\beta)}$ is again a torus, or the action is not free and properly discontinuous.

We focus on the linear maps with $\omega \neq 1$.

The maps $f_\omega : \mathbb{T} \circlearrowleft$ have fixed points: if $\omega^m = 1$ (m-roots of unity primitive) the

$$\# \mathrm{Fix}(f_\omega) = \begin{cases} 4 & \text{if } m = 2 \\ 3 & \text{"} \quad 3 \\ 2 & \text{"} \quad 4 \\ 1 & \text{"} \quad 6 \end{cases}$$

In particular the action of f is not free.

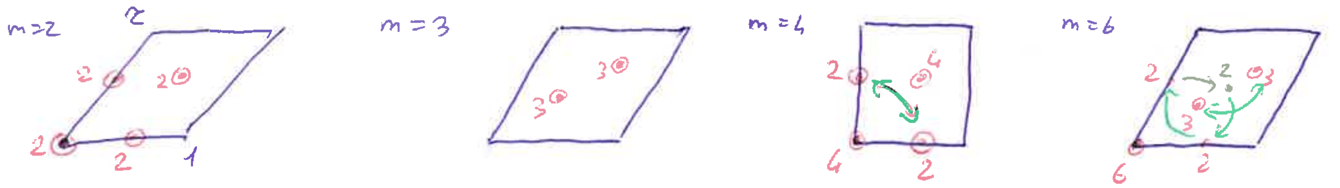
Nevertheless, we can define on the quotient a structure of Riemann surface.

Focus in the example $m=2$. The lattice is $\Lambda = \langle 1, \tau \rangle$, and the four fixed points are $0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}$ (mod Λ).

In each of such points η, we may consider local parameters $(z-\eta)^2$.

~~The resulting~~ so The quotient map $P_\omega : \mathbb{T} \to \mathbb{T}/_{\langle f_\omega \rangle} =: X$ is a ramified covering.

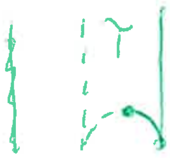
To understand what is X, we will use the Riemann-Hurwitz formula



Notice again that the case $m=2$ may happen for any lattice Λ, while 3, 4, 6 happen only for special cases.

In fact one can check that; since $\omega \in \Lambda$, $\omega = a + b\tau$, ~~and the exists a~~

We can pick τ in the "Segel domain" $\Upsilon = \left\{ |\tau| \geq 1,\ |\mathrm{Re}\,\tau| \leq \frac{1}{2},\ \underset{\mathrm{Im}\,\tau > 0}{} \ \mathrm{Re}\,\tau > 0 \text{ if either } \underset{|\tau| = 1 \text{ or } |\mathrm{Re}\,\tau| = \frac{1}{2}}{} \right\}$



And since $|\omega| = 1$, this implies that $\underset{m=4,6}{\omega = \tau}$, and one correspond to the drawings ↑ or $\omega = \tau^2$ $m=3$, $\tau = e^{\frac{2\pi i}{3}}$.

For $m=2$, the projection P_2 is known as the Weierstrass P-function (or a affine combination of it). $P(z) = \frac{1}{z^2} + \sum_{\substack{\lambda \in \Lambda \\ \lambda \neq 0}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right) =$

$= \sum_{\lambda \in \Lambda} \frac{1}{(z-\lambda)^2} - \sum_{\substack{\lambda \in \Lambda \\ \lambda \neq 0}} \frac{1}{\lambda^2}$ (which is clearly Λ-periodic).

For $m=3$, $\tau = e^{\frac{2\pi i}{3}}$ and $P_3 = P'$

Functions $\tilde{P} : \mathbb{T} \to \mathbb{C}$ are called (Weierstrass) elliptic functions.

For $m=4$, $\tau = i$ and $P_4 = P^2$

For $m=6$, $\tau = e^{\frac{2\pi i}{3}}$ and $P_6 = (P')^2 = P^3$.

Let now $\alpha \in \Lambda \cap \tau^{-1}(\Lambda)$, ~~with~~ of $|\alpha|^2 = d \geq 2$. Then $z \mapsto \alpha z$ defines a map f_α on $\mathbb{T} = \mathbb{C}/\Lambda$ ~~which is~~ of degree d which commutes with f_ω.

Hence it defines a holomorphic map $\underset{(\text{of } \deg f \geq 2)}{f : \mathbb{P}^1 \circlearrowleft}$.

Def: A rational map $f : \mathbb{P}^1 \circlearrowleft$ is a Lattès map if there exists a finite ramified covering $P : \mathbb{T} \to \mathbb{P}^1$, $\mathbb{T} = \mathbb{C}/\Lambda$ torus, and a map f_α (of degree $|\alpha|^2 \geq 2$) so that

$$\begin{array}{ccc} \mathbb{T} & \xrightarrow{f_\alpha} & \mathbb{T} \\ \downarrow P & & \downarrow P \\ \mathbb{P}^1 & \xrightarrow{f} & \mathbb{P}^1 \end{array}$$

$f \circ P = P \circ f_\alpha$.

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Prop: let $f:\mathbb{P}^1 \circlearrowleft$ be a Lattès map associated to $f_\alpha: \mathbb{T} \circlearrowleft$.
Then $\deg f = |\alpha|^2$ and $J(f) = \mathbb{P}^1$.

Proof: If we compute the number of preimages of a generic point (or of any point with multiplicity) ~~through~~ $f \circ P = P \circ f_\alpha$, we get $\deg f \cdot \deg P = \deg P \cdot \deg f_\alpha$ $\Rightarrow$ $\deg f = \deg f_\alpha = |\alpha|^2$.

Being P a ramified covering, outside the finite set $\mathcal{C}_P$ of critical points of P, we have that $P' \neq 0$. Hence: $\forall\, w \in \mathbb{T}$ periodic of period q, we have

$$\underset{\substack{\shortparallel \\ P(w)}}{f^q(z)} = P(f_\alpha^q(w)) = P(w) = z,$$ hence periodic points for f_α are sent to periodic points for f. Up to replace with an iterate, take $q=1$.

$$f'(z)\cdot P'(w) = P'(w)\cdot f_\alpha'(w) \underset{\substack{\uparrow \\ w \notin \mathcal{C}_{f_\alpha}}}{\Rightarrow} f'(z) = \alpha.$$

Since f_α-periodic repelling points are dense in $\mathbb{T}$, so is for f, and $J(f) = \hat{\mathbb{C}}$. □

The previous proof also tells us that critical points of f must be ~~[illegible]~~ ramification (i.e. critical values) for P

Proposition: ~~[illegible]~~ Lattès maps $f:\mathbb{P}^1 \circlearrowleft$ are strictly post-critically finite.

Def: let $f: \hat{\mathbb{C}} \circlearrowleft$ be a rational map. Denote by $\mathcal{C}_f$ the critical set of f.
The post-critical set $\underset{(PCF)}{PC(f)}$ is: $PC(f) = \bigcup_{n \geq 1} f^n(\mathcal{C}_f)$.
f is post-critically finite if. $PC(f)$ is a finite set, i.e., every critical point is preperiodic.
f is strictly PCF if every critical point is preperiodic but not periodic.

Proof (of prop.): ~~Being~~ ~~[illegible]~~ $\mathcal{C}_P$ the set of branching points of P.

We proceed as before, replacing the derivative with the local degree = ramification index e_w. Since $e_w(f_\alpha) = 1$ $\forall w \in T$, we get.

$$e_w(P) \cdot e_z(f) = e_{f_\alpha(w)}(P) \cdot 1. \qquad z = P(w)$$

In particular, $f_\alpha(\mathcal{C}_P) \subseteq \mathcal{C}_P$. and $f(\mathcal{C}_f) \subseteq P(\mathcal{C}_P)$

It follows that $PC(f) \subseteq P(\mathcal{C}_P)$ which is a finite set

A critical point cannot be periodic, or we would have a superattracting basin, against $J(f) = \hat{\mathbb{C}}$. □

Lattès maps can be divided into two classes:

<u>Flexible</u> Lattès maps, if we may vary Λ and L continuously so to obtain other Lattès maps f_ε not conformally conjugate to f

Rigid Lattès maps, where this is not possible.

Notice that to vary Λ continuously, $L = L_\alpha$ must be an integer multiplication (non CM tori are dense).

One can show that $\Lambda = \langle 1, \tau \rangle$ and $L = L_\alpha$ $\alpha \in \mathbb{Z}$ produce Flexible Lattès maps.

Remarks:

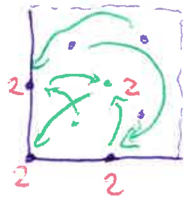
- We will see that $f: \hat{\mathbb{C}} \circlearrowleft$ strictly P.C.F, $\not\exists$ attracted by repelling cycles. $\Rightarrow J(f) = \hat{\mathbb{C}}$.
- There are other maps $f: \hat{\mathbb{C}} \circlearrowleft$ with this property and not Lattès.
- There are also maps not PCF for which $J(f) = \hat{\mathbb{C}}$; examples are given by Lyubich
- One can repeat the analogous construction for $\Lambda = \langle 1 \rangle \cong \mathbb{Z}$, and $\mathbb{C}/\mathbb{Z} \simeq \mathbb{C}^*$. In this case the maps obtained are powers, with $J(f) = \partial \mathbb{D}$, or Chebyshev polynomials, with $J(f) = [-2, 2]$. Again, there are other maps for which

$\mathcal{J}(P)$ is a $\partial \mathbb{D}$ or an interval.

For example: any composition of powers and elements in $\mathrm{Aut}(\mathbb{D})$.

See [Milnor, 2004] for further details.

Explicit examples.



- $\Lambda = \langle 1, i \rangle$, $\omega = -1$, $\alpha = 1+i \Rightarrow d = 2$.

From the formula $e_z(P) \cdot e_\omega(f) = e_{f_\alpha(z)}(P)$,

we infer that $e_f = \{ P(\underbrace{\tfrac{1}{4}(1+i)}_{c_1}), P(\underbrace{\tfrac{1}{4}(1+3i)}_{c_2}) \}$, and we have:

$$\begin{array}{l} c_1 \to e_1 \searrow \\ \qquad\qquad e_3 \to \infty \circlearrowleft \\ c_2 \to e_2 \nearrow \end{array}$$

(we set $P(0) = \infty$)

Up to change of coordinates, we may assume $e_3 = 0$, and

$$f(z) = \frac{az^2 + bz + c}{z} = az + b + \frac{c}{z}.$$

By linear change of coordinate $z \mapsto \lambda z$, we may assume $a = c$.

Then $f'(z) = a - \dfrac{a}{z^2}$ and $c_1 = 1$, $c_2 = -1$.

Hence we have $f(z) = a(z + \frac{1}{z}) + b$, $\quad f(+1) = 2a + b$
$\qquad f(-1) = -2a + b$

Notice that at the point $0 \in \mathbb{T}$, the projection P has order 2, i.e., locally $P(z) = z^2$.

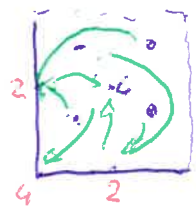
It follows that at ∞ f is of the form: $f(z) \approx \lambda z$, $f_\alpha(z) = \alpha z$, and,

$$f(P(z)) \sim P(f_\alpha(z)) \iff \lambda z^2 \sim \alpha^2 z^2.$$

Hence the multiplier of ∞ is $\lambda = \alpha^2 = 2i$, and $a = \frac{1}{\lambda} = -\frac{i}{2}$.

Hence $f(\pm 1) = b \mp i$
By direct computation, $f^{-1}(0) = -ib \pm \sqrt{-b^2 - 1}$ $\Big\} \Rightarrow b = 0$.

$\Rightarrow f(z) = -\frac{i}{2}(z + \frac{1}{z})$



- Similar computations but for the quotient by $\boxed{\omega = i}$

$\left[\underset{-1}{c_1} \overset{2}{\mapsto} \underset{1}{c_2} \overset{2}{\mapsto} \underset{0}{e_3} \mapsto \underset{\infty}{e_4} \circlearrowleft \right.$ $\qquad f(z) = -\frac{1}{4}(z + \frac{1}{z}) + \frac{1}{2}$.

- Flexible example: $\Lambda = \langle 1, \tau \rangle$ $\omega = -1$, $\alpha = 2 \Rightarrow d = 4$ $\qquad f(z) = \dfrac{4z(1-z)(1-k^2 z)}{(1 - k^2 z^2)^2}$

$k \in \mathbb{C} \setminus \{0, 1\}$ $\quad (k = k(\tau))$ $\quad$ See [Milnor, 2004]